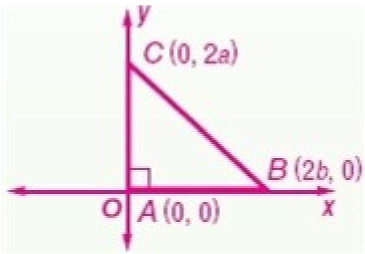
4-8 Triangles and Coordinate Proof

Position and label each triangle on the coordinate plane.

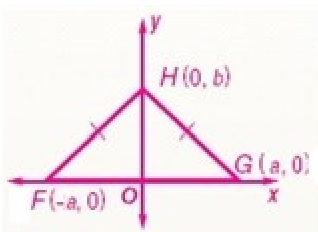
1. right ΔABC with legs $\overline{AC}$ and $\overline{AB}$ so that $\overline{AC}$ is $2a$ units long and leg $\overline{AB}$ is $2b$ units long

ANSWER:

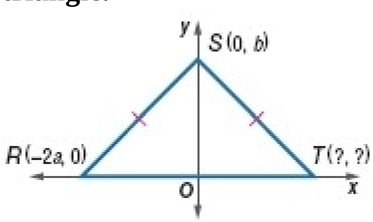


2. isosceles ΔFGH with base $\overline{FG}$ that is $2a$ units long

ANSWER:



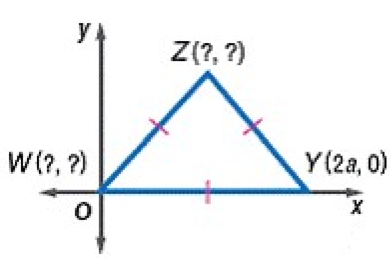
Name the missing coordinate(s) of each triangle.



3.

ANSWER:

$T(2a, 0)$

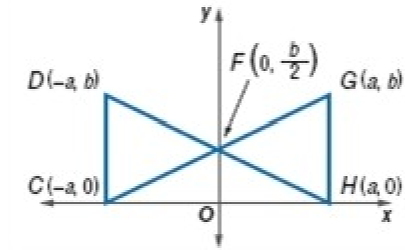


4.

ANSWER:

$W(0, 0)$, $Z\left(a, \sqrt{3}a\right)$

5. **JUSTIFY ARGUMENTS** Write a coordinate proof to show that $\Delta FGH \cong \Delta FDC$.



ANSWER:

$DC = \sqrt{(-a - (-a))^2 + (b-0)^2}$ or b

$GH = \sqrt{(a-a)^2 + (b-0)^2}$ or b

Since $DC = GH$, $\overline{DC} \cong \overline{GH}$.

$DF = \sqrt{(0+a)^2 + \left(\frac{b}{2} - b\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

$GF = \sqrt{(a-0)^2 + \left(b - \frac{b}{2}\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

$CF = \sqrt{(0+a)^2 + \left(\frac{b}{2} - 0\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

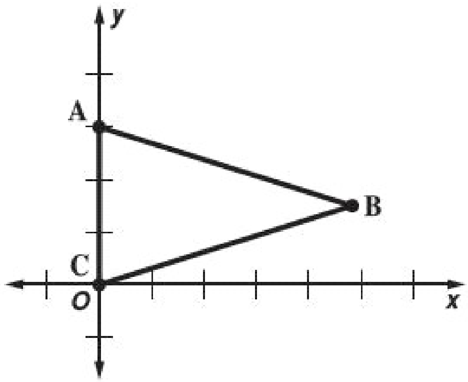
$HF = \sqrt{(a-0)^2 + \left(0 - \frac{b}{2}\right)^2}$ or $\sqrt{a^2 + \frac{b^2}{4}}$

Since $DF = GF = CF = HF$,
$\overline{DF} \cong \overline{GF} \cong \overline{CF} \cong \overline{HF}$. $\Delta FGH \cong \Delta FDC$ by SSS.

6. **FLAGS** Write a coordinate proof to prove that the large triangle in the center of the flag is isosceles. The dimensions of the flag are 3 feet by 5 feet and point B of the triangle bisects the right side of the flag.

ANSWER:

Given: $\triangle ABC$



Prove: $\triangle ABC$ is isosceles.

Proof: Use the Distance Formula to find *AB* and *BC*.

$AB = \sqrt{(5-0)^2 + (1.5-3)^2}$ or $\sqrt{27.25}$ or 5.2

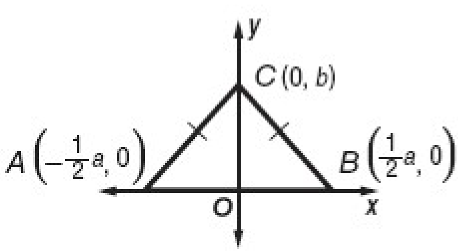
$BC = \sqrt{(5-0)^2 + (1.5-0)^2}$ or $\sqrt{27.25}$ or 5.2

Since *AB* = *BC*, $\overline{AB} \cong \overline{BC}$. Since the legs are congruent, $\triangle ABC$ is isosceles.

Position and label each triangle on the coordinate plane.

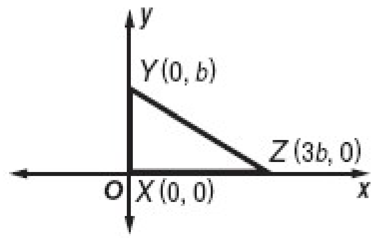
7. isosceles $\triangle ABC$ with base $\overline{AB}$ that is *a* units long

ANSWER:



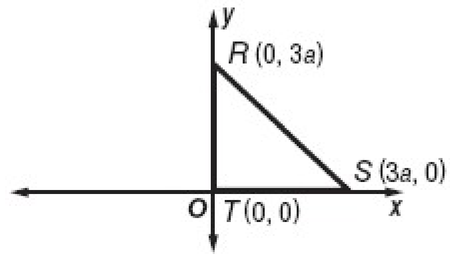
8. right $\triangle XYZ$ with hypotenuse $\overline{YZ}$, the length of $\overline{XY}$ is *b* units long, and the length of $\overline{XZ}$ is three times the length of $\overline{XY}$

ANSWER:



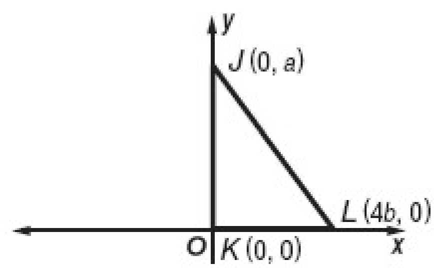
9. isosceles right $\triangle RST$ with hypotenuse $\overline{RS}$ and legs 3*a* units long

ANSWER:



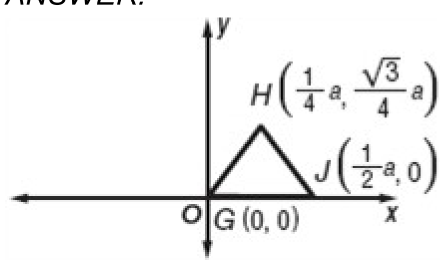
10. right $\triangle JKL$ with legs $\overline{JK}$ and $\overline{KL}$ so that $\overline{JK}$ is *a* units long and leg $\overline{KL}$ is 4*b* units long

ANSWER:



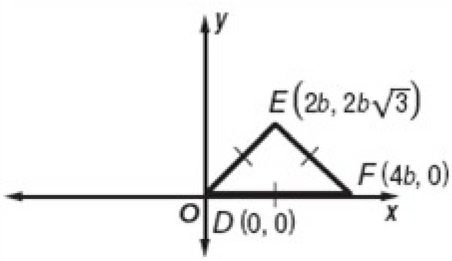
11. equilateral $\triangle GHJ$ with sides $\frac{1}{2}a$ units long

ANSWER:

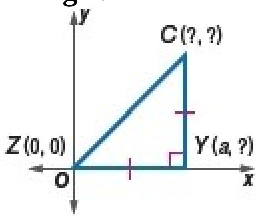


12. equilateral ΔDEF with sides 4*b* units long

ANSWER:

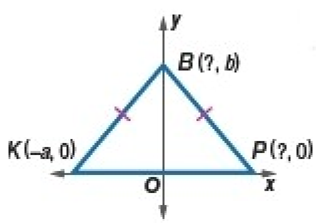


Name the missing coordinate(s) of each triangle.

13. 

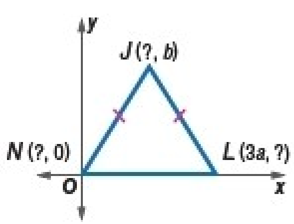
ANSWER:

C(*a*, *a*), *Y*(*a*, 0)

14. 

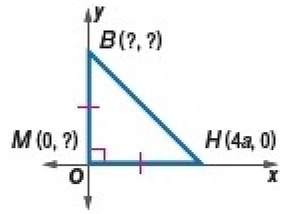
ANSWER:

P(*a*, 0), *B*(0, *b*)

15. 

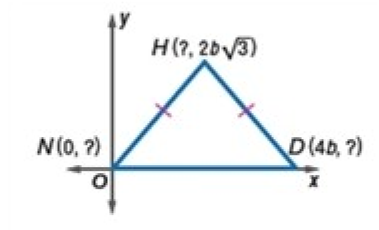
ANSWER:

N(0, 0), *J*(1.5*a*, *b*), *L*(3*a*, 0)

16. 

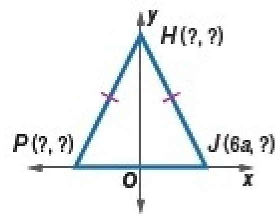
ANSWER:

M(0, 0), *B*(0, 4*a*)

17. 

ANSWER:

H(2*b*, 2*b* $\sqrt{3}$), *N*(0, 0), *D*(4*b*, 0)

18. 

ANSWER:

P(–6*a*, 0), *H*(0, *b*), *J*(6*a*, 0)

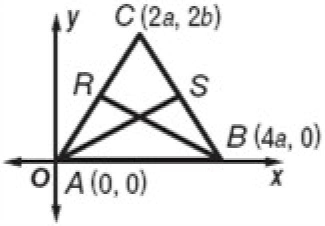
JUSTIFY ARGUMENTS Write a coordinate proof for each statement.

19. The segments joining the base vertices to the midpoints of the legs of an isosceles triangle are congruent.

ANSWER:

Given: Isosceles $\triangle ABC$ with $\overline{AC} \cong \overline{BC}$;

R and S are midpoints of legs $\overline{AC}$ and $\overline{BC}$.



Prove: $\overline{AS} \cong \overline{BR}$

Proof:

The coordinates of S are $\left(\frac{2a+4a}{2}, \frac{2b+0}{2}\right)$ or $(3a, b)$.

The coordinates of R are $\left(\frac{2a+0}{2}, \frac{2b+0}{2}\right)$ or (a, b).

$AS = \sqrt{(3a-0)^2 + (b-0)^2}$ or $\sqrt{9a^2 + b^2}$

$BR = \sqrt{(4a-a)^2 + (0-b)^2}$ or $\sqrt{9a^2 + b^2}$

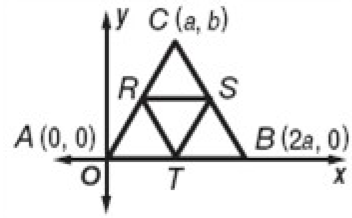
Since $AS = BR$, $\overline{AS} \cong \overline{BR}$.

20. The three segments joining the midpoints of the sides of an isosceles triangle form another isosceles triangle.

ANSWER:

Given: Isosceles triangle ABC; $\overline{BC} \cong \overline{AC}$;

R, S, and T are midpoints of their respective sides.



Prove: $\triangle RST$ is isosceles.

Proof:

Midpoint R is $\left(\frac{a+0}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{a}{2}, \frac{b}{2}\right)$.

Midpoint S is $\left(\frac{a+2a}{2}, \frac{b+0}{2}\right)$ or $\left(\frac{3a}{2}, \frac{b}{2}\right)$.

Midpoint T is $\left(\frac{2a+0}{2}, \frac{0+0}{2}\right)$ or $(a, 0)$

$RT = \sqrt{\left(\frac{a}{2}-a\right)^2 + \left(\frac{b}{2}-0\right)^2}$ or $\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

$ST = \sqrt{\left(\frac{3a}{2}-a\right)^2 + \left(\frac{b}{2}-0\right)^2}$ or $\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}$

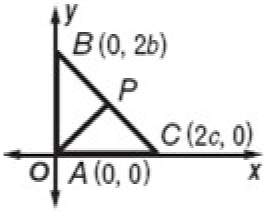
$RT = ST$ and $\overline{RT} \cong \overline{ST}$ and $\triangle RST$ is isosceles.

PROOF Write a coordinate proof for each statement.

21. The measure of the segment that joins the vertex of the right angle in a right triangle to the midpoint of the hypotenuse is one-half the measure of the hypotenuse.

ANSWER:

Given: Right ΔABC with right $\angle BAC$; P is the midpoint of $\overline{BC}$.



Prove: $AP = \frac{1}{2}BC$

Proof:

Midpoint P is $\left(\frac{0+2c}{2}, \frac{2b+0}{2}\right)$ or (c, b).

$AP = \sqrt{(c-0)^2+(b-0)^2}$ or $\sqrt{c^2+b^2}$

$BC = \sqrt{(2c-0)^2+(0-2b)^2} = \sqrt{4c^2+4b^2}$ or $2\sqrt{c^2+b^2}$

$\frac{1}{2}BC = \sqrt{c^2+b^2}$

So, $AP = \frac{1}{2}BC.$

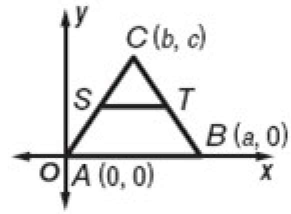
22. If a line segment joins the midpoints of two sides of a triangle, then its length is equal to one half the length of the third side.

ANSWER:

Given: ΔABC

S is the midpoint of $\overline{AC}$.

T is the midpoint of $\overline{BC}$.



Prove: $ST = \frac{1}{2}AB$

Proof:

The coordinates of S are $\left(\frac{b}{2}, \frac{c}{2}\right)$ and the coordinates of T are $\left(\frac{a+b}{2}, \frac{c}{2}\right)$

$ST = \sqrt{\left(\frac{a+b}{2}-\frac{b}{2}\right)^2+\left(\frac{c}{2}-\frac{c}{2}\right)^2}$ or $\frac{a}{2}$

$AB = \sqrt{(a-0)^2+(0-0)^2}$ or a

$ST = \frac{1}{2}AB$

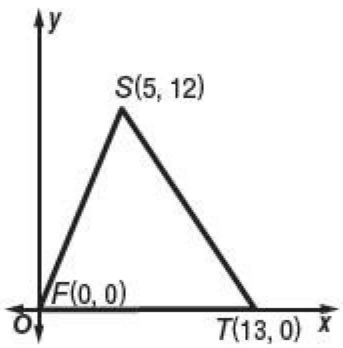
23. **RESEARCH TRIANGLE** The cities of Raleigh, Durham, and Chapel Hill, North Carolina, form what is known as the Research Triangle. The approximate latitude and longitude of Raleigh are 35.82°N 78.64°W, of Durham are 35.99°N 78.91°W, and of Chapel Hill are 35.92°N 79.04°W. Show that the triangle formed by these three cities is scalene.

ANSWER:

The distance between Raleigh and Durham is about 0.32 units, between Raleigh and Chapel Hill is about 0.41 units, and between Durham and Chapel Hill is about 0.15 units. Since none of these distances are the same, the Research Triangle is scalene.

24. **PARTY PLANNING** Three friends live in houses with backyards adjacent to a neighborhood bike path. They decide to have a round-robin party using their three homes, inviting their friends to start at one house and then move to each of the other two. If one friend's house is centered at the origin, then the location of the other homes are (5, 12) and (13, 0). Write a coordinate proof to prove that the triangle formed by these three homes is isosceles.

ANSWER:



$FS = \sqrt{(5-0)^2 + (12-0)^2}$ or 13
$FT = \sqrt{(13-0)^2 + (0-0)^2}$ or 13

Since the distance between the first house at (0, 0) and the third house at (13, 0) is the same as the distance between the first house at (0, 0) and the second house at (5, 12), the triangle formed by the three homes is isosceles.

Draw ΔXYZ and find the slope of each side of the triangle. Determine whether the triangle is a right triangle. Explain

25. $X(0, 0)$, $Y(2h, 2h)$, $Z(4h, 0)$

ANSWER:

slope of $\overline{XY}$ = 1, slope of $\overline{YZ}$ = –1, slope of $\overline{ZX}$ = 0; since 1(–1) = –1, $\overline{XY} \perp \overline{YZ}$. Therefore, ΔXYZ is a right triangle.

26. $X(0, 0)$, $Y(1, h)$, $Z(2h, 0)$

ANSWER:

slope of $\overline{XY}$ = h, slope of $\overline{YZ} = \frac{h}{1-2h}$, slope of $\overline{ZX}$ = 0; When $h = \frac{1}{2}$, $\overline{YZ}$ is a vertical segment and $\overline{XZ} \perp \overline{YZ}$. When h = 1, $h \cdot \frac{h}{1-2h}$ = -1 and $\overline{XY} \perp \overline{YZ}$. So ΔXYZ is only a right triangle if $h = \frac{1}{2}$ or 1.

27. **MULTI-STEP** Two high school clubs have gone camping. Club A pitches their tent 25 miles north of the ranger's station. Club B wants to set up their tent so that it is 9 miles north of the ranger's station and forms a right triangle with the ranger's station and Club A's tent.
a. Where should Club B set up camp?
b. What assumptions did you make? Write a coordinate proof to prove that the figure formed is a right triangle.

ANSWER:

a. Club B should set up camp at (12, 9).
b. I assumed that the ranger's station was located at the origin and that Club B set up camp east of Club A and the ranger's station. To determine where Club B should set up camp, we need to determine the slopes of the lines connecting their camp to Club A's camp and their camp to the ranger's station.
Slope between Club A's tent and Club B's tent = $\frac{25-9}{0-x} = -\frac{16}{x}$
Slope between Club B's tent and ranger's station = $\frac{9-0}{x-0} = \frac{9}{x}$
For the legs of the triangle to form a right angle, the slopes must have a product of –1.

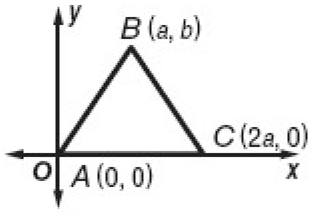
$$-1 = -\frac{16}{x} \cdot \frac{9}{x}$$
$$-1 = \frac{-144}{x^2}$$
$$x^2 = 144$$
$$x = \pm 12$$

Since I assumed that Club B camped east of Club A, their coordinates are (12, 9).

28. **PROOF** Write a coordinate proof to prove that ΔABC is an isosceles triangle if the vertices are $A(0, 0)$, $B(a, b)$, and $C(2a, 0)$.

ANSWER:

Given: vertices $A(0, 0)$, $B(a, b)$, and $C(2a, 0)$



Prove: ΔABC is isosceles.

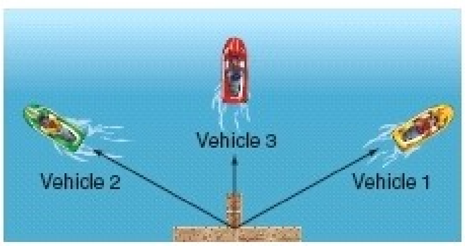
Proof:

$AB = \sqrt{(a-0)^2+(b-0)^2}$ or $\sqrt{a^2+b^2}$

$BC = \sqrt{(2a-a)^2+(0-b)^2}$ or $\sqrt{a^2+b^2}$

Since, $AB = BC$, $\overline{AB} \cong \overline{BC}$. So, ΔABC is isosceles.

29. **WATER SPORTS** Three personal watercraft vehicles launch from the same dock. The first vehicle leaves the dock traveling due northeast, while the second vehicle travels due northwest. Meanwhile, the third vehicle leaves the dock traveling due north.



The first and second vehicles stop about 300 yards from the dock, while the third stops about 212 yards from the dock.

a. If the dock is located at (0, 0), sketch a graph to represent this situation. What is the equation of the line along which the first vehicle lies? What is the equation of the line along which the second vehicle lies? Explain your reasoning.

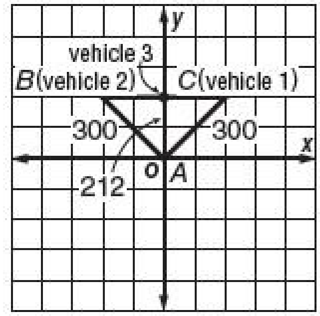
b. Write a coordinate proof to prove that the dock, the first vehicle, and the second vehicle form an isosceles right triangle.

c. Find the coordinates of the locations of all three watercrafts. Explain your reasoning.

d. Write a coordinate proof to prove that the positions of all three watercrafts are approximately collinear and that the third watercraft is at the midpoint between the other two.

ANSWER:

a.



The equation of the line along which the first vehicle lies is $y = x$. The slope is 1 because the vehicle travels the same number of units north as it does east of the origin and the y-intercept is 0. The equation of the line along which the second vehicle lies is $y = -x$. The slope is –1 because the vehicle travels the same number of units north as it does west of the origin and the y-intercept is 0.

b. The paths taken by both the first and second vehicles are 300 yards long. Therefore the paths are congruent. If two sides of a triangle are congruent, then the triangle is isosceles.

c. First vehicle, $(150\sqrt{2}, 150\sqrt{2})$; second vehicle, $(-150\sqrt{2}, 150\sqrt{2})$; third vehicle, (0, 212); the paths taken by the first two vehicles form the hypotenuse of a right triangle. Using the Pythagorean Theorem, the distance between the third vehicle and the first and second vehicle can be calculated. The third vehicle travels due north and therefore, remains on the y-axis.

d. The y-coordinates of the first two vehicles are $150\sqrt{2} \approx 212.13$, while the y-coordinate of the third vehicle is 212. Since all three vehicles have approximately the same y-coordinate, they are approximately collinear. The midpoint of the first and second vehicle is $\left(\frac{150\sqrt{2}-150\sqrt{2}}{2}, \frac{212+212}{2}\right)$ or (0, 212), the location of the third vehicle.

30. **ANALYZE RELATIONSHIPS** The midpoints of the sides of a triangle are located at $(a, 0)$, $(2a, b)$ and (a, b). If one vertex is located at the origin, what are the coordinates of the other vertices? Explain your reasoning.

ANSWER:
$(2a, 0)$, $(2a, 2b)$; Using the Midpoint Formula,
$(a, 0) = \left(\frac{0+x_2}{2}, \frac{0+y_2}{2}\right)$, so
$x_2 = 2a$ and $y_2 = 0$.
$(a, b) = \left(\frac{0+x_2}{2}, \frac{0+y_2}{2}\right)$, so
$x_2 = 2a$ and $y_2 = 2b$.

ANALYZE RELATIONSHIPS Find the coordinates of point *L* so ΔJKL is the indicated type of triangle. Point *J* has coordinates (0, 0) and point *K* has coordinates $(2a, 2b)$.

31. scalene triangle

ANSWER:
Sample answer: $(a, 0)$

32. right triangle

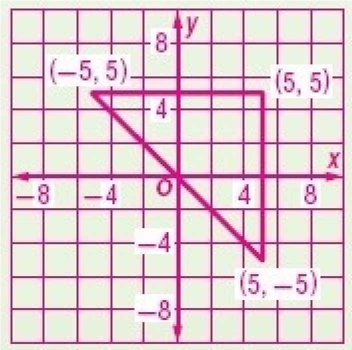
ANSWER:
Sample answer: $(2a, 0)$ or $(0, 2b)$

33. isosceles triangle

ANSWER:
Sample answer: $(4a, 0)$

34. **ORGANIZE IDEAS** Draw an isosceles right triangle on the coordinate plane so that the midpoint of its hypotenuse is the origin. Label the coordinates of each vertex.

ANSWER:
Sample answer:

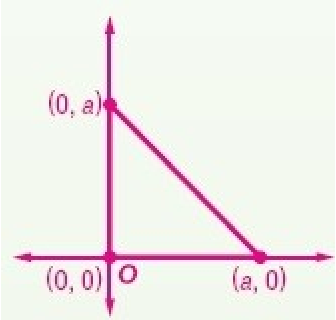


35. **JUSTIFY ARGUMENTS** Use a coordinate proof to show that if you add *n* units to each *x*-coordinate of the vertices of a triangle and *m* to each *y*-coordinate, the resulting figure is congruent to the original triangle.

ANSWER:
Given: ΔABC with coordinates $A(0, 0)$, $B(a, b)$, and $C(c, d)$ and ΔDEF with coordinates $D(0 + n, 0 + m)$, $E(a + n, b + m)$, and $F(c + n, d + m)$.
Prove: $\Delta DEF \cong \Delta ABC$
Proof:
$AB = \sqrt{(a-0)^2 + (b-0)^2}$ or $\sqrt{a^2 + b^2}$
$DE = \sqrt{(a+n-(0+n))^2 + (b+m-(0+m))^2}$ or $\sqrt{a^2 + b^2}$
Since $AB = DE$, $\overline{AB} \cong \overline{DE}$.
$BC = \sqrt{(c-a)^2 + (d-b)^2}$ or $\sqrt{c^2 - 2ac + a^2 + d^2 - 2bd + b^2}$
$EF = \sqrt{(c+n-(a+n))^2 + (d+m-(b+m))^2}$ or $\sqrt{c^2 - 2ac + a^2 + d^2 - 2bd + b^2}$
Since $BC = EF$, $\overline{BC} \cong \overline{EF}$.
$CA = \sqrt{(c-0)^2 + (d-0)^2}$ or $\sqrt{c^2 + d^2}$
$FD = \sqrt{(0+n-(c+n))^2 + (0+m-(d+m))^2}$ or $\sqrt{c^2 + d^2}$
Since $CA = FD$, $\overline{CA} \cong \overline{FD}$.
Therefore, $\Delta DEF \cong \Delta ABC$ by the SSS Postulate.

36. **ANALYZE RELATIONSHIPS** A triangle has vertex coordinates (0, 0) and $(a, 0)$. If the coordinates of the third vertex are in terms of *a*, and the triangle is isosceles, identify the coordinates and position the triangle on the coordinate plane.

ANSWER:
Sample answer: $(0, a)$;



37. **WRITING IN MATH** Explain why following each guideline below for placing a triangle on the coordinate plane is helpful in proving coordinate proofs.

a. Use the origin as a vertex of the triangle.

b. Place at least one side of the triangle on the x- or y-axis.

c. Keep the triangle within the first quadrant if possible.

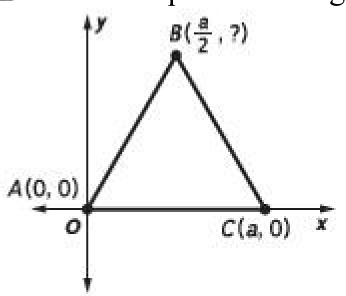
ANSWER:

a. Using the origin as a vertex of the triangle makes calculations easier because the coordinates are (0, 0).

b. Placing at least one side of the triangle on the x- or y-axis makes it easier to calculate the length of the side since one of the coordinates will be 0.

c. Keeping a triangle within the first quadrant makes all of the coordinates positive, and makes the calculations easier.

38. $\triangle ACD$ is an equilateral triangle.



What is the y-coordinate of point A?

A $\frac{a}{2}$

B $\frac{\sqrt{3}a}{2}$

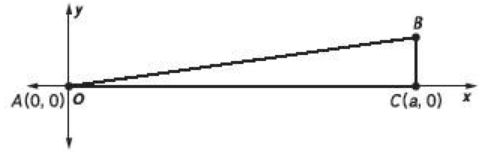
C a

D $\frac{3a}{2}$

ANSWER:

B

39. **ACT/SAT** The figure shown models a ramp with a height of 12 inches.



What is the slope of the ramp?

F 0

G $\frac{12}{a}$

H 12

J $12a$

ANSWER:

G